Int J Theor Phys (2008) 47: 380–392
DOI 10.1007/s10773-007-9460-9

Energetics of the Einstein–Rosen Spacetime

L. Herrera · A. Di Prisco · J. Carot · N.O. Santos

Received: 10 March 2007 / Accepted: 8 June 2007 / Published online: 18 July 2007

Abstract A study covering some aspects of the Einstein–Rosen metric is presented. The electric and magnetic parts of the Weyl tensor are calculated. It is shown that there are no purely magnetic E-R spacetimes, and also that a purely electric E-R spacetime is necessarily static. The geodesics equations are found and circular ones are analyzed in detail. The super-Poynting and the "Lagrangian" Poynting vectors are calculated and their expressions are found for two specific examples. It is shown that for a pulse-type solution, both expressions describe an inward radially directed flow of energy, far behind the wave front. The physical significance of such an effect is discussed.

L. Herrera (✉) · A. Di Prisco
Escuela de Física, Facultad de Ciencias, Universidad Central de Venezuela, Caracas, Venezuela
e-mail: laherrera@cantv.net.ve

A. Di Prisco
e-mail: adiprisc@fisica.ciens.ucv.ve

J. Carot
Departament de Física, Universitat Illes Balears, 07122 Palma de Mallorca, Spain
e-mail: jcarot@uib.es

N.O. Santos
School of Mathematical Sciences, Queen Mary, University of London, London, UK
e-mail: N.O.Santos@qmul.ac.uk

N.O. Santos
Laboratório Nacional de Computação Científica, 25651-070 Petrópolis RJ, Brazil

N.O. Santos
LERMA/CNRS-FRE 2460, Université Pierre et Marie Curie, ERGA, Boîte 142, 4 Place Jussieu, 75005 Paris Cedex 05, France
e-mail: santos@ccr.jussieu.fr

1 Introduction

Einstein–Rosen (E-R) spacetime [1] has attracted the attention of researchers for many years (see [2–12] and references therein). Its interest may be understood by recalling that E-R represents the simplest example of a spacetime describing outgoing gravitational waves.

It is our purpose in this work to study some aspects of the E-R spacetime which have not been considered until now, and which may shed some light on the nature of such a physically meaningful spacetime.

Particular attention deserves the question of whether or not cylindrical gravitational waves have energy and momentum. Indeed, this problem has been investigated by many authors (see [13–16] and references therein) using different energy-momentum pseudotensors. However such a practice has been questioned by many physicists, for the obvious reason that those objects are not tensors (see [17]). Fortunately though, we have available a tensor quantity which allows to define a covariant (super)-energy density and a (super)-Poynting vector, namely the Bel–Robinson tensor [18–20]. We shall study the energetics of E-R spacetime with the help of these quantities.

With this goal in mind, we shall first calculate the electric $E_{\alpha\beta}$ and magnetic $H_{\alpha\beta}$ parts of Weyl tensor, as well as the two invariants obtained from them, for the Einstein–Rosen metric. These geometric objects have been under discussion for many years (see [18–38] and references therein). Not only because of the eventual relationship of the magnetic part of the Weyl tensor with rotation [21, 28], but also because of its link with gravitational radiation [18–20, 24, 27, 29–31, 39].

From the expressions obtained for the above objects, it follows (although the presented proof is somehow restricted) that the vanishing of the magnetic part implies that the spacetime is static, whereas the vanishing of the electric part of the Weyl tensor is shown to imply that the spacetime is Minkowski, in agreement with the fact that purely magnetic vacuum space-times do no exist [25, 26, 28, 36–38].

Next we obtain the geodesic equations for a test particle in an E-R spacetime. We analyze the circular motion and compare the behavior of the test particle in the static case with respect to its behavior in two (non-static) solutions of the E-R family, namely: a pulse-type solution [12, 40] and the Kramer solution [41] (both in a static background).

The motivation for doing so become clear when the resulting differences (with respect to the static situation) are analyzed and interpreted with the help of the expression of the radial component of the super-Poynting vector [42], and the Poynting (pseudo)-vector defined by Stephani [43].

In the first example (pulse-type solution) the angular velocity (far behind the front) is smaller than in the static case, this difference decreases asymptotically as the system gets back to the static situation. The physical interpretation of this result will be given in terms of the energetics of the system (see the last section).

In the case of the Kramer solution (in a static background), the time average of the tangential velocity of the particle is greater than in the static case. This suggests an increasing of the gravitational mass of the source, which might be related to an inward radially directed flow of energy. However, the time average of the Stephani vector vanishes, and, unfortunately, it was impossible to evaluate the time average of the super-Poynting vector, even with the help of a computer, although it seems that such an average is different from zero. An alternative interpretation, not involving an inward flux of energy, is also presented.

At this point it is worth mentioning that in a general time-dependent E-R spacetime, specific constraints have to be satisfied, for circular geodesics to exist (see Sect. 5.1 below). Accordingly, in the two examples examined here, we shall define the conditions under which, circular geodesics do in fact exist. Of course all the discussion will be carried on, upon satisfaction of the above mentioned conditions. Also it is worth stressing that both solutions considered here are defined in a static background.

Thus, in the case of the pulse (in a static background) we shall see that conditions for the existence of circular orbits are satisfied, behind the pulse, very far from the front (in Sect. 5.1 we specify what we mean by “very far”) where the energy flux (as defined by the super-Poynting or the Stephani vector) is negative (inward).

It is worth noticing that this last result is not in contradiction with the fact that the rate of change of C-energy is shown to be negative in [5], since that proof is only valid for large values of r, where in fact the radial component of, both, the super-Poynting and the Stephani vectors, are positive (see Sect. 6).

In the case of the Kramer spacetime (in a static background), conditions for the existence of circular geodesics are satisfied only for the time average of the metric.

2 The Einstein–Rosen metric and Its Electric and Magnetic Weyl Tensor

The line element reads

$$ds^2 = -e^{2\gamma-2\psi}(dt^2 - dr^2) + e^{2\psi}dz^2 + r^2e^{-2\psi}d\phi^2 \tag{1}$$

where $\psi = \psi(t,r)$ and $\gamma = \gamma(t,r)$ satisfy the Einstein equations:

$$\psi_{tt} - \psi_{rr} - \frac{\psi_r}{r} = 0, \tag{2}$$

$$\gamma_t = 2r\psi_r\psi_t, \tag{3}$$

$$\gamma_r = r(\psi_r^2 + \psi_t^2) \tag{4}$$

where indexes stand for differentiation with respect to t and r.

Now, for an observer at rest in the frame of (1), the four-velocity vector has components

$$u^\alpha = (e^{\psi-\gamma}, 0, 0, 0) \tag{5}$$

and it is obvious that for such a congruence of observers the vorticity $\omega_{\alpha\beta}$ vanishes.

2.1 The Electric and Magnetic Parts of Weyl Tensor

The electric and magnetic parts of Weyl tensor, $E_{\alpha\beta}$ and $H_{\alpha\beta}$, respectively, are formed from the Weyl tensor $C_{\alpha\beta\gamma\delta}$ and its dual $\tilde{C}_{\alpha\beta\gamma\delta}$ by contraction with the four-velocity vector given by (5):

$$E_{\alpha\beta} = C_{\alpha\gamma\beta\delta}u^\gamma u^\delta, \tag{6}$$

$$H_{\alpha\beta} = \tilde{C}_{\alpha\gamma\beta\delta}u^\gamma u^\delta = \frac{1}{2}\epsilon_{\alpha\gamma\epsilon\delta}C^{\epsilon\delta}{}_{\beta\rho}u^\gamma u^\rho, \quad \epsilon_{\alpha\beta\gamma\delta} \equiv \sqrt{-g}\,\eta_{\alpha\beta\gamma\delta} \tag{7}$$

where $\eta_{\alpha\beta\gamma\delta} = +1$ for $\alpha, \beta, \gamma, \delta$ in even order, -1 for $\alpha, \beta, \gamma, \delta$ in odd order and 0 otherwise.

We have calculated the magnetic and electric part using GRTensor. Thus, one has for the components of the magnetic Weyl tensor:

$$H_{23} = H_{32} = re^{2(\psi-\gamma)}\psi_t\left[-3\psi_r - \frac{\psi_{tr}}{\psi_t} + r(\psi_t^2 + 3\psi_r^2)\right] \tag{8}$$

and the rest of the components being zero.

Regarding the electric part, one gets:

$$E_{11} = \psi_t^2 - \psi_r^2 + \frac{1}{r}\psi_r, \tag{9}$$

$$E_{22} = e^{(4\psi-2\gamma)}\left[-2\psi_t^2 - \psi_r^2 - \psi_{rr} - \frac{\psi_r}{r} + r\psi_r(3\psi_t^2 + \psi_r^2)\right], \tag{10}$$

$$E_{33} = r^2 e^{-2\gamma}[\psi_{rr} - r\psi_r(3\psi_t^2 + \psi_r^2) + \psi_t^2 + 2\psi_r^2]. \tag{11}$$

Notice, however, that these components are not all independent, since they satisfy the relation

$$E_{22}e^{2\gamma-4\psi} + \frac{E_{33}}{r^2}e^{2\gamma} = -E_{11}. \tag{12}$$

The two invariants

$$Q = H_b^a E_a^b, \qquad I = E_b^a E_a^b - H_b^a H_a^b$$

can now be easily computed to give:

$$Q = 0, \tag{13}$$

$$\begin{aligned} I = -\frac{2}{r^2}e^{4(\psi-\gamma)}\{&(\psi_t - \psi_r)^3(\psi_t + \psi_r)^3 r^4 \\ &+ r^3[-2\psi_t\psi_{tr}(3\psi_r^2 + \psi_t^2) + 3\psi_r^5 + (2\psi_{rr} - 6\psi_t^2)\psi_r^3 + 3\psi_r\psi_t^2(\psi_t^2 + 2\psi_{rr})] \\ &+ r^2[\psi_{tr}^2 + 6\psi_{tr}\psi_t\psi_r - 2\psi_r^4 + 9(\psi_t^2 - 3\psi_{rr})\psi_r^2 - \psi_{rr}^2 - 3\psi_t^2(\psi_t^2 + \psi_{rr})] \\ &- r\psi_r(3\psi_t^2 + \psi_{rr}) - \psi_r^2\}. \end{aligned} \tag{14}$$

3 Purely Magnetic Solutions (PMS)

Let us now see that there exist no purely magnetic solutions, i.e. solutions satisfying $E_{\alpha\beta} = 0$, in agreement with the known fact that purely magnetic vacua with a non-rotating congruence are flat indeed.

From (9–11) one obtains

$$6\psi_r^2 + \psi_{rr} - \frac{\psi_r}{r} - 4r\psi_r^3 = 0 \tag{15}$$

whose first general integral is

$$\psi_r = \frac{1}{2r} \pm \frac{1}{2r\sqrt{1-4r^2c}} \tag{16}$$

with c an arbitrary function of time, but since the range of r extends to infinity, it turns out that $c = 0$.

Thus, either $\psi_r = 0$ which implies Minkowski, or

$$\psi = \ln r + \beta \tag{17}$$

where β is an arbitrary function of time.

However, from $E_{11} = 0$ and (17) it follows that β is a constant, then implying that $\psi_t = 0$ and hence $H_{\alpha\beta} = 0$, which produces a Minkowski spacetime. Accordingly, we conclude that, as expected, there are no PMS Einstein–Rosen waves.

4 Purely Electric Solutions (PES)

Let us now see if there exist solutions satisfying the condition $H_{\alpha\beta} = 0$.

It is clear from (8) that $\psi_t = 0$ satisfies the condition $H_{\alpha\beta} = 0$. The question is whether or not there exist solutions with $\psi_t \neq 0$ also satisfying $H_{\alpha\beta} = 0$. We shall see that this is not the case.

Indeed, the general form of ψ, satisfying the wave equation (2) (see [2, 44]), with the outgoing wave condition (i.e. no inward traveling waves), is

$$\psi = \mathrm{Re}\int_0^\infty A(\omega)e^{-i\omega t}H_0^{(1)}(\omega r)d\omega \tag{18}$$

where $H_0^{(1)}(\omega r)$ is the Hankel function of the first kind. Then it follows

$$\psi_t = \mathrm{Re}\int_0^\infty -i\omega A(\omega)e^{-i\omega t}H_0^{(1)}(\omega r)d\omega, \tag{19}$$

$$\psi_r = \mathrm{Re}\int_0^\infty -\omega A(\omega)e^{-i\omega t}H_1^{(1)}(\omega r)d\omega \tag{20}$$

and

$$\psi_{tr} = \mathrm{Re}\int_0^\infty i\omega^2 A(\omega)e^{i\omega t}H_1^{(1)}(\omega r)d\omega \tag{21}$$

where it has been used the fact that [45]

$$\frac{d}{dz}[z^{-\nu}H_\nu^{(p)}(z)] = -z^{-\nu}H_{\nu+1}^{(p)}(z). \tag{22}$$

Next, let us consider the low frequency regime ($\omega \lesssim \frac{1}{r}$). From the fact that ψ is always finite and $\int_0^\infty x^n H_0^{(1)}(x)dx$ converges only for $n > -1$, it follows that as $\omega \to 0$, $A(\omega) \sim \omega^n$ with $n > -1$. This implies that the leading terms in low-frequency ($\omega \lesssim \frac{1}{r}$) contributions to ψ_t, ψ_r and ψ_{tr} are

$$\psi_t\left(\omega \lesssim \frac{1}{r}\right) \sim \psi_r\left(\omega \lesssim \frac{1}{r}\right) \sim \frac{1}{r^{2+n}} \tag{23}$$

and

$$\psi_{tr}\left(\omega \lesssim \frac{1}{r}\right) \sim \frac{1}{r^{3+n}}. \tag{24}$$

From the above it is clear that the leading contribution in (8), in the low frequency regime, comes from the term ψ_{tr}, therefore the condition $H_{\alpha\beta} = 0$ would imply $\psi_{tr} = 0$. Since this is clearly incompatible with the general solution (18), we have to assume $\psi_t = 0$.

Even though the consistency of any solution must be assured for any value of ωr (including the low frequency regime), it may be argued however that the proof is limited to the low frequency regime and therefore is not completely general.

So, let us present an alternative argument. Let us suppose that $A(\omega)$ has a compact support, i.e. $A = 0$ for $\omega > \omega_c$, where ω_c is some constant. Then close to the symmetry axis where $\omega r \ll 1$ we have (see [44, 45])

$$H_0^{(1)}(\omega r) \approx -\frac{2i}{\pi} \ln \frac{2}{\omega r} \tag{25}$$

and

$$H_1^{(1)}(\omega r) \approx -\frac{2i}{\pi} \frac{\Gamma(1)}{\omega r}. \tag{26}$$

Feeding back (25) and (26) into (8) it follows at once that condition $H_{\alpha\beta} = 0$ implies $\psi_t = 0$.

Thus, we may say that all PES are necessarily static. In agreement with the well known fact that all static solutions are purely electric.

5 The Geodesics

The geodesic equations may be obtained from the Lagrangian

$$\mathcal{L} = \frac{1}{2} g_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} \tag{27}$$

then the Euler–Lagrange equations

$$\frac{d}{ds}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^\alpha}\right) - \frac{\partial \mathcal{L}}{\partial x^\alpha} = 0 \tag{28}$$

for the metric (1) (dots denote derivatives with respect to s) become

$$\begin{aligned} &e^{2(\gamma-\psi)}[(\gamma_t - \psi_t)(\dot{t}^2 + \dot{r}^2) + 2(\gamma_r - \psi_r)\dot{t}\dot{r} + \ddot{t}] \\ &\quad + e^{2\psi}\psi_t \dot{z}^2 - e^{-2\psi} r^2 \psi_t \dot{\phi}^2 = 0, \end{aligned} \tag{29}$$

$$\begin{aligned} &e^{2(\gamma-\psi)}[2\dot{t}\dot{r}(\gamma_t - \psi_t) + (\dot{t}^2 + \dot{r}^2)(\gamma_r - \psi_r) + \ddot{r}] \\ &\quad - e^{2\psi}\psi_r \dot{z}^2 + e^{-2\psi}(r^2\psi_r - r)\dot{\phi}^2 = 0, \end{aligned} \tag{30}$$

$$\ddot{z} + 2\dot{z}\dot{t}\psi_t + 2\dot{z}\dot{r}\psi_r = 0, \tag{31}$$

$$\ddot{\phi} + 2\dot{\phi}\frac{\dot{r}}{r} - 2\dot{r}\dot{\phi}\psi_r - 2\dot{t}\dot{\phi}\psi_t = 0 \tag{32}$$

furthermore from (1) we obtain

$$-\epsilon = -e^{2\gamma-2\psi}(\dot{t}^2 - \dot{r}^2) + e^{2\psi}\dot{z}^2 + r^2 e^{-2\psi}\dot{\phi}^2 \tag{33}$$

where $\epsilon = 0$, 1 or -1 if the geodesics are, respectively, null, timelike, or space-like.

Observe that (32) may be write as

$$\frac{d}{ds}\left(r^2 e^{-2\psi}\frac{d\phi}{ds}\right) = 0, \tag{34}$$

implying that $L = r^2 e^{-2\psi}\frac{d\phi}{ds}$ is a constant of motion.

5.1 Circular Geodesics

Now we restrict ourselves to the case of circular geodesics, implying

$$\dot{r} = \dot{z} = 0. \tag{35}$$

Then using (35) in (29–33) we obtain for the angular velocity

$$\omega = \frac{\dot{\phi}}{\dot{t}}, \tag{36}$$

$$\omega^2 = -\frac{e^{2\gamma}(\gamma_r - \psi_r)}{r^2\psi_r - r} \tag{37}$$

or using the field equation (4)

$$\omega^2 = \frac{e^{2\gamma}\psi_t^2}{1 - r\psi_r} - e^{2\gamma}\frac{\psi_r}{r}. \tag{38}$$

From the definition for tangential velocity

$$V^\mu = (-g_{00})^{-1/2} U^\mu \tag{39}$$

where

$$U^\mu = (\delta^\mu_\alpha + u^\mu u_\alpha)\frac{dx^\alpha}{dt} \tag{40}$$

and u^μ is given by (5), we obtain

$$V^\mu = e^{-(\gamma-\psi)}(0, 0, 0, \omega) \tag{41}$$

thus

$$V^\mu V_\mu = e^{-2\gamma} r^2 \omega^2 \tag{42}$$

which, using (38) can be written as

$$V^\mu V_\mu = \frac{(r\psi_t)^2}{(1 - r\psi_r)} - r\psi_r. \tag{43}$$

Next, substituting (34) into (33) and using (31) we obtain for circular orbits

$$L^2 = -\epsilon r^3 e^{-2\psi} \times \frac{r\psi_{,t}{}^2 + r\psi_{,r}{}^2 - \psi_{,r}}{r^2\psi_{,t}{}^2 + r^2\psi_{,r}{}^2 - 1}. \quad (44)$$

The above equation is a necessary condition for the existence of the circular orbit. Since the left hand side of the above equation is constant, the right hand side should be independent on the time t too. This specific constraint should be satisfied in order to assure the existence of circular geodesics in any E-R spacetime.

This is indeed the case, for the two examples examined here. Thus, in the case of the pulse (in a static background), very far from the front ($t \gg r$) we shall obtain (see below)

$$L^2 \approx L_{LC}^2 + O\left(\frac{r}{t}\right) \quad (45)$$

where L_{LC} correspond to the static (Levi-Civita) solution. Thus, whenever we can neglect terms of the order $O(\frac{r}{t})$ and higher, we can safely assume that circular gedodesics do exist.

In the case of the Kramer spacetime (in a static background), we shall consider only the time average of the corresponding variables, in which case is obvious that the requirement above on L is also satisfied.

5.2 Static Case

This case is represented by the well known Levi-Civita solution [46–48]

$$\psi_{LC} = \alpha - \beta \ln r, \quad \alpha, \beta \text{ constants} \quad (46)$$

and

$$\gamma_{LC} = \beta^2 \ln r. \quad (47)$$

Using (46) into (38) and (43) we obtain ω^2 and $V^\mu V_\mu$ for this case

$$\omega^2 = \beta r^{2(\beta^2-1)}, \quad (48)$$

$$V^\mu V_\mu = \beta. \quad (49)$$

5.3 A Pulse Solution in a Static Background

Let us now consider a cylindrical source which is static for a period of time until it starts contracting and emits a sharp pulse of radiation traveling outward from the axis. Then, the function ψ can be written as [12, 40]

$$\psi = \frac{1}{2\pi} \int_{-\infty}^{t-r} \frac{f(t')dt'}{[(t-t')^2 - r^2]^{1/2}} + \psi_{LC}. \quad (50)$$

In (50) ψ_{LC} represents the Levi-Civita static solution (46), and $f(t)$ is a function of time representing the strength of the source of the wave and it is assumed to be of the form

$$f(t) = f_0 \delta(t), \quad (51)$$

where f_0 is a constant and $\delta(t)$ is the Dirac delta function. It can be shown that (50) satisfies the wave equation (2). Then we get

$$\psi = \psi_{LC}, \quad t < r; \tag{52}$$

$$\psi = \frac{f_0}{2\pi(t^2 - r^2)^{1/2}} + \psi_{LC}, \quad t > r, \tag{53}$$

from the above, (45) can be easily deduced.

The function ψ, as well as its derivatives, is regular everywhere except at the wave front determined by the surface $t = r$, followed by a tail decreasing with t.

From (53) and Einstein's equations, we obtain

$$\gamma(r,t) = \beta^2 \ln r + \left(\frac{f_0}{\pi}\right)^2 \frac{r^2}{8(t^2-r^2)^2} - \frac{f_0}{\pi}\frac{\beta}{(t^2-r^2)^{1/2}}. \tag{54}$$

Using (53) into (38) and (43) we obtain

$$\omega^2 = e^{2\gamma}\left[\frac{\beta}{r^2} - \frac{f_0}{2\pi}\frac{1}{(t^2-r^2)^{3/2}} + \left(\frac{f_0}{\pi}\right)^2 \frac{t^2}{4(t^2-r^2)^3}\left(1+\beta-\frac{f_0}{2\pi}\frac{r^2}{(t^2-r^2)^{3/2}}\right)^{-1}\right] \tag{55}$$

with γ given by (54).

In the limit $t \gg r$, neglecting terms of order $O(\frac{r}{t})$ and higher we obtain for ω^2

$$\omega^2 = r^{2(\beta^2-1)}\beta e^{-2\beta f_0/\pi t} = \omega_{LC}^2 e^{-2\beta f_0/\pi t}. \tag{56}$$

From the above, it follows that the angular velocity is smaller than in the static case.

Finally, it is worth noticing that just behind the front, in the limit $t \lesssim r$, the leading term in (14) vanishes identically, supporting the link between gravitational radiation and the vanishing of Q and I [27].

5.4 The Kramer Solution in a Static Background

Let us now consider the Kramer solution [41] in a static background [43], thus we have.

$$\begin{aligned}
\psi &= -\beta \ln r + C J_0(r)\cos t, \\
\gamma &= \beta^2 \ln r + \frac{1}{2}C^2 r\{r[J_0^2(r) + J_1^2(r)] - 2J_0(r)J_1(r)\cos^2 t\} \\
&\quad -2C\beta J_0(r)\cos t
\end{aligned} \tag{57}$$

then from (38) and (57) we obtain

$$\omega^2 = r^{2\beta^2}\exp[Cr^2\{r(J_0^2(r)+J_1^2(r)) - 2J_0(r)J_1(r)\cos^2 t\} - 4C\beta J_0(r)\cos t]\left[\frac{\beta}{r^2} + \frac{CJ_1(r)\cos t}{r} + \frac{C^2 J_0^2(r)\sin^2 t}{1+\beta+CrJ_1(r)\cos t}\right] \tag{58}$$

and from (43) and (57)

$$V^{\mu}V_{\mu} = \beta + CrJ_1\cos t + \frac{r^2C^2J_0^2(r)\sin^2 t}{1+\beta+CrJ_1(r)\cos t}. \tag{59}$$

The time average for (59) turns out to be:

$$\langle V^{\mu}V_{\mu}\rangle = \beta + (1+\beta)\frac{J_0^2(r)}{J_1^2(r)}\left[1-\left(1-\frac{r^2C^2J_1^2(r)}{(1+\beta)^2}\right)^{1/2}\right]. \tag{60}$$

The second term in the right hand side of (60) being positive, it is clear that this time average is larger than the corresponding value in the static case.

6 Super-Poynting and Lagrangian Poynting Vectors in the Einstein–Rosen Spacetime

The super-Poynting vector as defined in [42] is given by

$$P_{\alpha} = \epsilon_{\alpha\beta\gamma\delta}E^{\beta}_{\rho}H^{\gamma\rho}u^{\delta} \tag{61}$$

giving

$$P_2 = P_3 = 0 \tag{62}$$

and

$$P_1 = \frac{H_{23}e^{\psi-\gamma}\sqrt{-g}}{r^2}\left(E_{33}\frac{e^{2\psi}}{r^2} - e^{-2\psi}E_{22}\right) \tag{63}$$

or, using (8–11)

$$\begin{aligned} P_1 = e^{3(\psi-\gamma)}\Big[&\psi_{rr}(-6\psi_r\psi_t + 2r\psi_t^3 + 6r\psi_t\psi_r^2) - 2\psi_{rr}\psi_{tr} \\ &+ \psi_{tr}\left(-3\psi_t^2 - 3\psi_r^2 - \frac{\psi_r}{r} + 6r\psi_r\psi_t^2 + 2r\psi_r^3\right) - 8\psi_t^3\psi_r \\ &+ 3r\psi_t^5 + 30r\psi_t^3\psi_r^2 - 6\psi_r^3\psi_t - \frac{3\psi_r^2\psi_t}{r} + 15r\psi_r^4\psi_t \\ &- 6r^2\psi_r\psi_t^5 - 20r^2\psi_t^3\psi_r^3 - 6r^2\psi_r^5\psi_t\Big]. \end{aligned} \tag{64}$$

An alternative expression to evaluate the flux of gravitational energy is given in [43]. This Poynting vector, named "Lagrangian" by Stephani, yields for the radial component of the energy flux:

$$S^1 = -2r\psi_r\psi_t. \tag{65}$$

It can be checked without difficulty that (65) is essentially equivalent to the expressions obtained from the Landau–Lifshitz [49] and Tolman [50] energy-momentum complex, when calculated in Cartesian coordinates (see equations (5), (10) and (14) in [16]).

For the pulse-type solution, it follows from (53)

$$S^1 = \frac{f_0 t}{\pi(t^2-r^2)^{3/2}}\left[\frac{f_0 r^2}{2\pi(t^2-r^2)^{3/2}} - \beta\right] \tag{66}$$

which in the limit $t \gg r$ (far behind the pulse) becomes

$$S^1 \approx -\frac{f_0 \beta}{\pi t^2} \tag{67}$$

where we have neglected terms of order $O(\frac{r}{t})$ and higher. This is clearly a negative quantity.

However, just behind the pulse ($t \approx r, t > r$), the flux is positive (directed outward) if only

$$\beta < \frac{f_0 r^2}{2\pi (t^2 - r^2)^{3/2}}. \tag{68}$$

Next, taking the limit $t \gg r$ in the radial component of the super-Poynting vector we get

$$P^1 \approx e^{3(\psi-\gamma)}\left(-\frac{3 f_0}{2\pi t^2 r^3}\right)(\beta^2 + 4\beta^3 + 5\beta^4 + 2\beta^5) \tag{69}$$

where we have neglected terms of order $O(\frac{r}{t})$ and higher. This is also a negative quantity.

On the other hand, in the limit $t \approx r (t > r)$ (just behind the pulse) we obtain

$$P^1 \approx \frac{e^{3(\psi-\gamma)} f_0^5 t^5}{2\pi^5 (t^2 - r^2)^{15/2}}\left[\frac{f_0 t^2}{\pi (t^2 - r^2)^{3/2}} - \frac{7\beta}{2}\right] \tag{70}$$

which is positive if

$$\beta < \frac{2 f_0 r^2}{7\pi (t^2 - r^2)^{3/2}}. \tag{71}$$

Therefore it is apparent from the above and (45), that circular geodesics in the pulse-type solution, exist only when the radial flux of energy (as measured by either the super-Poynting or the "Lagrangian" vector) is directed inward.

Unfortunately we were unable to calculate the time average of (64) for the Kramer solution, since the resulting integrals cannot be expressed in terms of elementary functions. From a very rough numerical estimate it seems however, that such average is not vanishing.

In the next section we shall present a discussion on the results obtained so far.

7 Discussion

We have seen that E-R waves produce a non-vanishing magnetic Weyl tensor. This fact together with the vanishing of the vorticity for the congruence of observers at rest in (1) reinforces the link between gravitational radiation and the magnetic part of Weyl tensor. We have seen further, that no purely magnetic E-R waves exist, as expected for a purely magnetic vacua, with a non-rotating congruence. Also, it was shown that purely electric solutions of this type are necessarily static. Proving thereby that a E-R spacetime is a PES if and only if it is static.

Next we have written down the geodesic equations and exhibited the influence of the gravitational wave on the circular motion of a test particle, by comparing with the static case.

In the first example (pulse-type solution) the angular velocity is smaller than in the static case, this difference decreases asymptotically as the system gets back to the static situation.

The physical interpretation of this result is straightforward: we have an initially static system, whose mass per unit of length is related to β [48], next the source emits a wave front traveling outwards, carrying out energy.

This explains why the angular velocity of the particle is smaller than in the static case. After that, the spacetime behind the front tends asymptotically to a static situation with the same β as initially. This explains why the exponent in (56) decreases with time. On the other hand it should be clear that in order to restore the static situation with the same initial β the system should absorb some energy, in order to compensate for the energy which has been radiated away.

This is indeed the picture which follows from the expressions for the radial component of the super-Poynting and the Stephani vectors. Although it is true that just behind the front, the energy flux is directed outward, it should be stressed that there are not circular geodesics there. These are allowed far from the front where terms of the order of $O(\frac{r}{t})$ and higher can be neglected.

In the case of the Kramer solution (in a static background), the time average of the tangential velocity of the particle is greater than in the static case. This suggests an increasing in the active gravitational mass of the source as compared with the static case. Such an effect might be interpreted in two different ways.

On the one hand, Kramer interprets his solution as describing a gravitational wave propagating in the z-direction. Then the energy of this wave could account for the increasing of the active gravitational mass of the source. Observe that the time average of the radial component of the Stephani vector vanishes in this case, which is consistent with the interpretation above.

On the other hand, it could be that, as in the pulse case, there is an incoming flow of energy which is responsible for the increasing of the total energy of the source. If this is so, then at least the time average of the radial component of the super-Poynting vector should be negative.

Unfortunately the expression is so complicated that we were unable to find such average, even with the help of a computer. However, if this last interpretation turns out to be correct, then it is clear that the super-Poynting vector would be more suitable for describing the flow of gravitational energy than the "Lagrangian" Poynting vector.

Finally, and just as a curiosity, it is worth mentioning that in classical electrodynamics, the Poynting vector for the field of a current along an infinite wire (with non-vanishing resistance), describes a flux of electromagnetic energy directed radially into the wire [51].

In this example, Feynman dismisses the significance of such an effect by arguing that it is deprived of any physical relevance.

In our case however, the presence of a radial, inwardly directed flux of gravitational energy (at least in the example of the pulse solution), is necessary in order to restore the energy carried away by the pulse, and, at the same time, to explain the time evolution pattern of an "observable" quantity such as the angular velocity of a test particle in a circular motion around the source.

Acknowledgements One of us (J.C.) gratefully acknowledges financial support from the Spanish Ministerio de Educación y Ciencia through the grant FPA2004-03666. L.H. wishes to thank FUNDACION POLAR for financial support and Universitat Illes Balears for financial support and hospitality. A.D.P. also acknowledges hospitality of the Physics Department of the Universitat Illes Balears.

References

1. Einstein, A., Rosen, N.: J. Franklin Inst. **223**, 43 (1937)

2. Thorne, K.S.: Phys. Rev. B **138**, 251 (1965)
3. Morgan, T.A.: Gen. Relativ. Gravit. **4**, 273 (1973)
4. Piran, T.: Phys. Rev. Lett. **41**, 1085 (1978)
5. Apostolatos, T.A., Thorne, K.S.: Phys. Rev. D **46**, 2435 (1992)
6. Echeverria, F.: Phys. Rev. D **47**, 2271 (1993)
7. Letelier, P.S., Wang, A.: Phys. Rev. D **49**, 5105 (1994)
8. Chiba, T.: Prog. Theor. Phys. **95**, 321 (1996)
9. Nolan, B.C.: Phys. Rev. D **65**, 104006 (2002)
10. Nakao, K., Morisawa, Y.: Class. Quantum Gravity **21**, 2101 (2004)
11. Nakao, K., Morisawa, Y.: gr-qc/0502017
12. Herrera, L., Santos, N.O.: Class. Quantum Gravity **22**, 2407 (2005)
13. Rosen, N.: Helv. Phys. Acta Suppl. **4**, 171 (1956)
14. Rosen, N.: Phys. Rev. **110**, 291 (1958)
15. Rosen, N., Virbhadra, K.S.: Gen. Relativ. Gravit. **26**, 429 (1993)
16. Virbhadra, K.S.: gr-qc/9509034
17. Chandrasekhar, S., Ferrari, V.: Proc. Roy Soc. Lond. A **435**, 645 (1991)
18. Bel, L.: C. R. Acad. Sci. **247**, 1094 (1958)
19. Bel, L.: Cah. Phys. **16**, 59 (1962)
20. Bel, L.: Gen. Relativ. Gravit. **32**, 2047 (2000)
21. Glass, E.N.: J. Math. Phys. **16**, 2361 (1975)
22. Barnes, A., Rowlingson, R.: Class. Quantum Gravity **6**, 949 (1989)
23. Matarrese, S., Pantano, O., Saez, D.: Phys. Rev. D **47**, 1311 (1993)
24. Bruni, M., Matarrese, S., Pantano, O.: Astrophys. J. **445**, 958 (1995)
25. McIntosh, C., Arianrhod, R., Wade, S., Hoenselaers, C.: Class. Quantum Gravity **11**, 1555 (1994)
26. Haddow, B.M.: J. Math. Phys. **18**, 1378 (1995)
27. Bonnor, W.B.: Class. Quantum Gravity **12**, 499 (1995)
28. Bonnor, W.B.: Class. Quantum Gravity **12**, 1483 (1995)
29. Dunsby, P.K.S., Basset, B.A., Ellis, G.F.R.: Class. Quantum Gravity **14**, 1215 (1997)
30. Maartens, R., Ellis, G.F.R., Siklos, T.: Class. Quantum Gravity **14**, 1927 (1997)
31. Hogan, P., Ellis, G.F.R.: Class. Quantum Gravity **14**, A171 (1997)
32. van Elst, H., Uggla, C., Lesame, W.M., Ellis, G.F.R., Maartens, R.: Class. Quantum Gravity **14**, 1151 (1997)
33. van Elst, H., Ellis, G.F.R.: Class. Quantum Gravity **15**, 3545 (1998)
34. Maartens, R., Lesame, W.M., Ellis, G.F.R.: Class. Quantum Gravity **15**, 1005 (1998)
35. Lozanowski, C., Aarons, M.: Class. Quantum Gravity **16**, 4075 (1999)
36. Van den Bergh, N.: Class. Quantum Gravity **20**, L1 (2003)
37. Van den Bergh, N.: Class. Quantum Gravity **20**, L165 (2003)
38. Ferrando, J., Saez, J.: Class. Quantum Gravity **20**, 2835 (2003)
39. Herrera, L., Santos, N.O., Carot, J.: gr-qc/0511112
40. Carmeli, M.: Classical Fields. General Relativity and Gauge Theory. Wiley, New York (1982)
41. Kramer, D.: Class. Quantum Gravity **16**, L75 (1999)
42. Maartens, R., Basset, B.A.: Class. Quantum Gravity **15**, 705 (1998)
43. Stephani, H.: Gen. Relativ. Gravit. **35**, 467 (2003)
44. Arfken, G.: Mathematical Methods for Physicists. Academic Press, New York (1970)
45. Lebedev, N.N.: Special Functions and Its Applications. Prentice Hall, Englewood Cliffs (1965)
46. Levi-Civita, T.: Rend. Acc. Lincei **28**, 101 (1919)
47. Stephani, H., Kramer, D., MacCallum, M., Honselaers, C., Herlt, E.: Exact Solutions to Einstein's Field Equations, 2nd edn. Cambridge University Press, Cambridge (2003)
48. Herrera, L., Santos, N.O.: J. Math. Phys. **39**, 3817 (1998)
49. Landau, L.D., Lifshitz, E.M.: The Classical Theory of Fields. Pergamon, London (1987)
50. Tolman, R.: Relativity Thermodynamics and Cosmology. Oxford University Press, London (1934)
51. Feynman, R.P., Leighton, R.B., Sand, M.: Lectures on Physics II, pp. 27–28. Addison–Wesley, Reading (1964)